Emergent Einstein Equation and p-adic Tensor Network

18th May, 2021,
8th International Conference on p-Adic Mathematical Physics and its Applications

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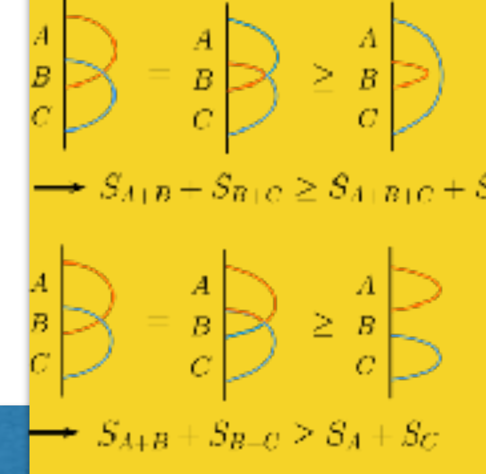
Work done in collaboration with :
Wei Li, Charles Melby Thompson JHEP 2019 (5), 1-35, JHEP 2019 (4), 170
Lin Chen, Xiong Liu
arXiv:2102.12022, 2102.12023, 2102.12024

Overview

- Tensor network and AdS/CFT
  - p-adic tensor network — Bending the BT tree
    - RG flow and emergent “Einstein equation”
  - Outlook

Many body entanglement and holographic theories

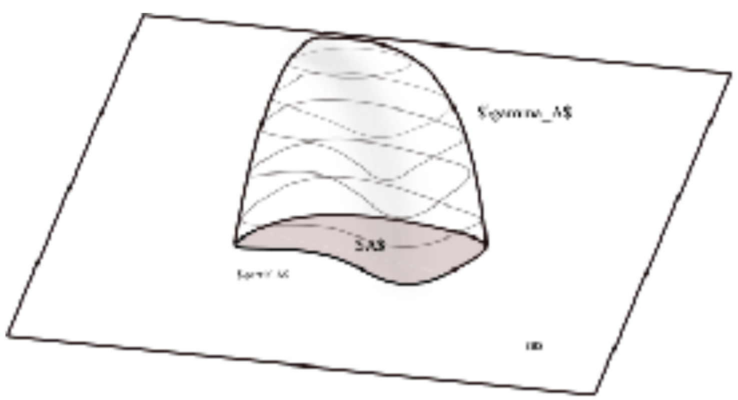


Quantum gravity …… algebra + geometry

Entanglement satisfies e.g. strong sub-additivity that looks like triangle inequalities — somehow they fit well with geometric data

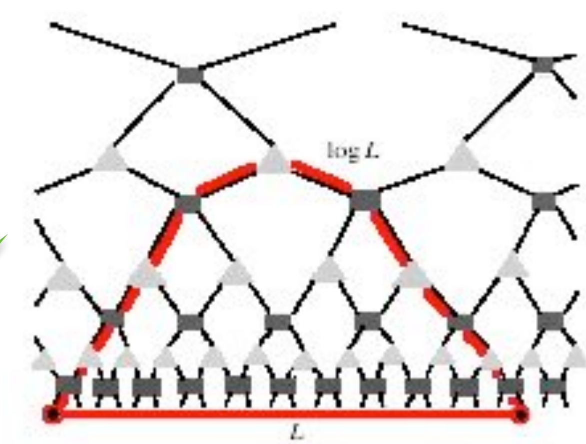
1. AdS/CFT says entanglement is geometry

Ryu-Takayanagi Formula:
(2006)



$$S_{EE} = \frac{A}{4G}$$

2. appropriate models that realise these ideas



Picture courtesy Orus

Brian Swingle (2012)

Tensor network is a geometrization of entanglement. It is explicitly local.

Quick detour 1 : AdS/CFT Dictionary

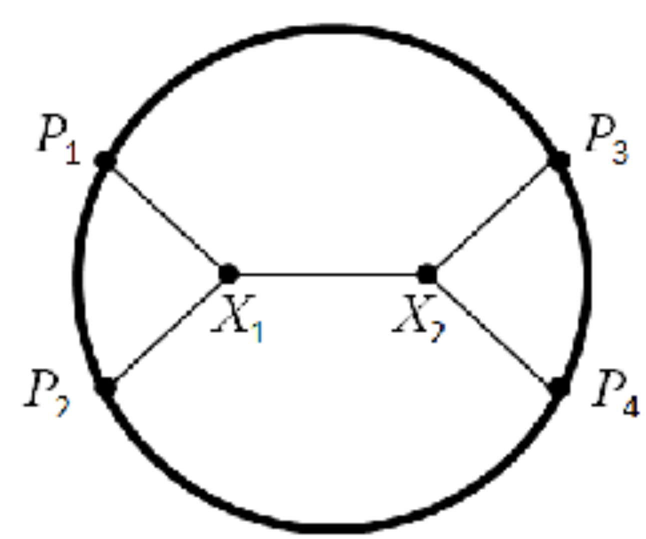
CFT	Gravity
Global symmetry: Conformal symmetry $SL(2,d)$ R symmetry $SO(6)$	Isometry of space Isometry of AdS_{d+1} $ds^2 = \frac{L^2}{z^2}(dz^2 + g_{\mu\nu}dx^\mu dx^\nu)$ S_5

CFT	Gravity
$g_{\rm YM}$	g_s
t'Hooft coupling $\lambda = g_{\rm YM}^2 N$	string length $\frac{L^4}{l_s^4}$
N^{-2}	Newton's constant G_N
$Z_{CFT}(\{J_i\})$	$Z_{\rm String}(\{J_i\}) \sim \int D[\phi, G_{\mu\nu}, A_\mu]\vert_{\{J_i\}_{z=0}} e^{-S_{Gr}+\cdots}$

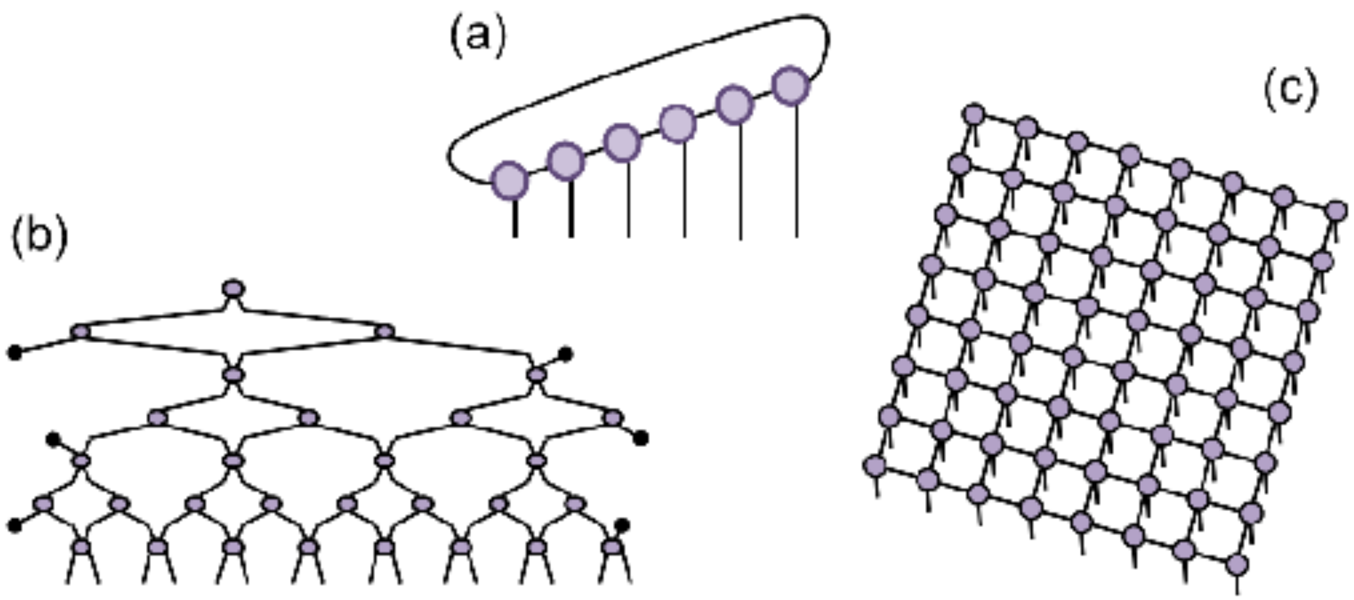
Strong weak duality

operators	fields
O_Δ Δ	$\phi_m(x,z)$ $\Delta = \frac{d}{2} + \sqrt{d^2/4 + m^2L^2}$
J_μ $T_{\mu\nu}$	A_μ $g_{\mu\nu}$

CFT correlation functions obtained from Witten diagrams of bulk fields



Quick detour 2— tensor networks



picture courtesy Orus

- Tensor network is a numerical method to obtain efficiently the ground states of given Hamiltonian

- $$|\psi\rangle = \sum f_{i_1,\cdots i_N} |i_1, \cdots i_N\rangle \qquad \leftarrow D^N$$

  $$= \sum T_{i_1 \mu\nu} T_{i_2 \nu\rho} T_{i_N \gamma\mu} |i_1, \cdots i_N\rangle \qquad \leftarrow N D^3$$

In this light what is “Einstein equation”?

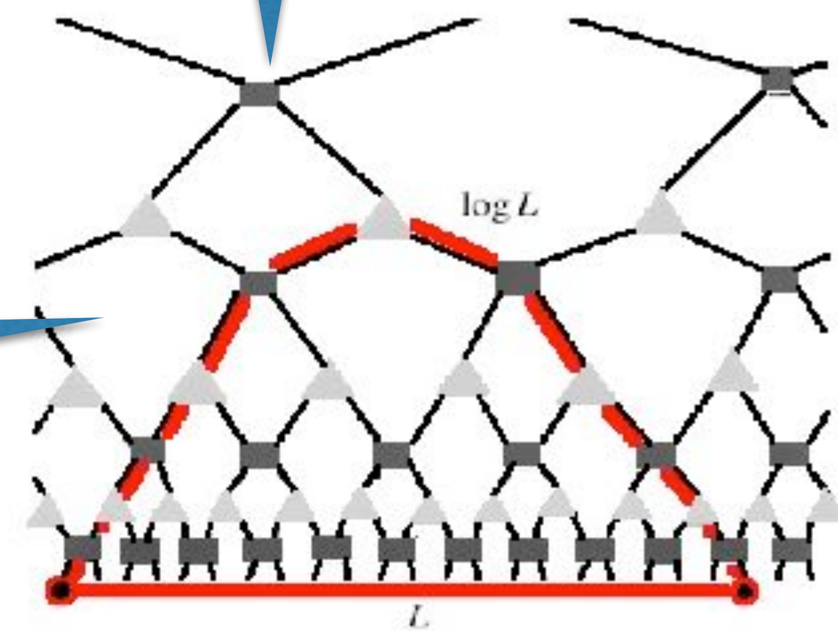
What tensors?

Einstein tensor

stress tensor in the *bulk*

- G = T

What graph?



Einstein equation : some sort of changes in matter excitations changes the overall entanglement of some states….. Problem is — we don’t know what is the Hilbert space of gravity…..

Bending the BT tree

One-page summary of p-adic CFT

CFT	p-adic CFT
$x \in \mathbb{R}$	$x \in \mathbb{Q}_p$
$x \to \frac{ax+b}{cx+d}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$	$x \to \frac{ax+b}{cx+d}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2,\mathbb{Q}_p)$
$\mathcal{O}_i(x) \to \lvert\frac{ad-bc}{(cx+d)^2}\rvert^{-\Delta_i}\mathcal{O}_i(x)$	$\mathcal{O}_i(x) \to \lvert\frac{ad-bc}{(cx+d)^2}\rvert_p^{-\Delta_i}\mathcal{O}_i(x)$
$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum C_{ij}^k \lvert x-y\rvert^{\Delta_k-\Delta_i-\Delta_j}\mathcal{O}_k(y) + \text{descendents}$	$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k C_{ij}^k \lvert x-y\rvert_p^{\Delta_k-\Delta_i-\Delta_j}\mathcal{O}_k(y)$

$$x = p^v\left(\sum_{m=0}^{\infty} a_m p^m\right)$$

$$|x|_p = p^{-v}$$

$$(x,y)_p = |x-y|_p$$

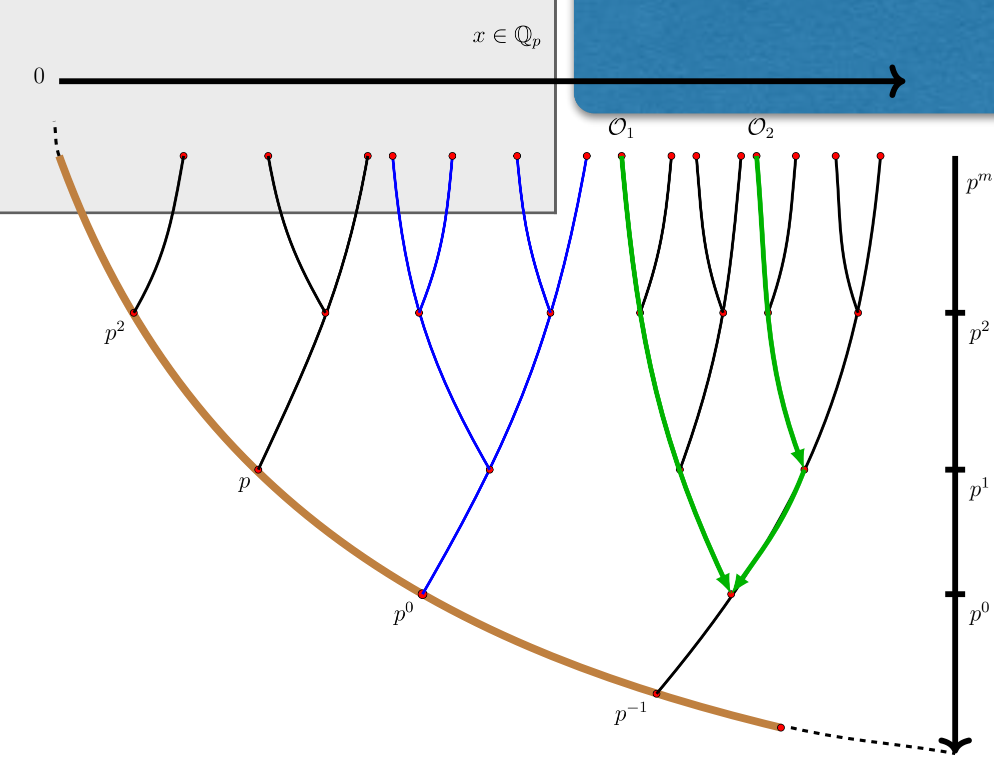
$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = C_i\delta_{ij}|x-y|_p^{-2\Delta_i} \qquad \langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}(z)_k\rangle = \frac{C_i C_{jk}^i}{|x-y|_p^{\Delta_{ij}}|x-z|_p^{\Delta_{ik}}|y-z|^{\Delta_{jk}}}$$

One line review of p-adic AdS/CFT:

Gubser et al. Commun.Math.Phys. 352 (2017) no.3, 1019-1059 ;
Heydeman, Matthew *et al.* Adv.Theor.Math.Phys. 22 (2018) 93-176

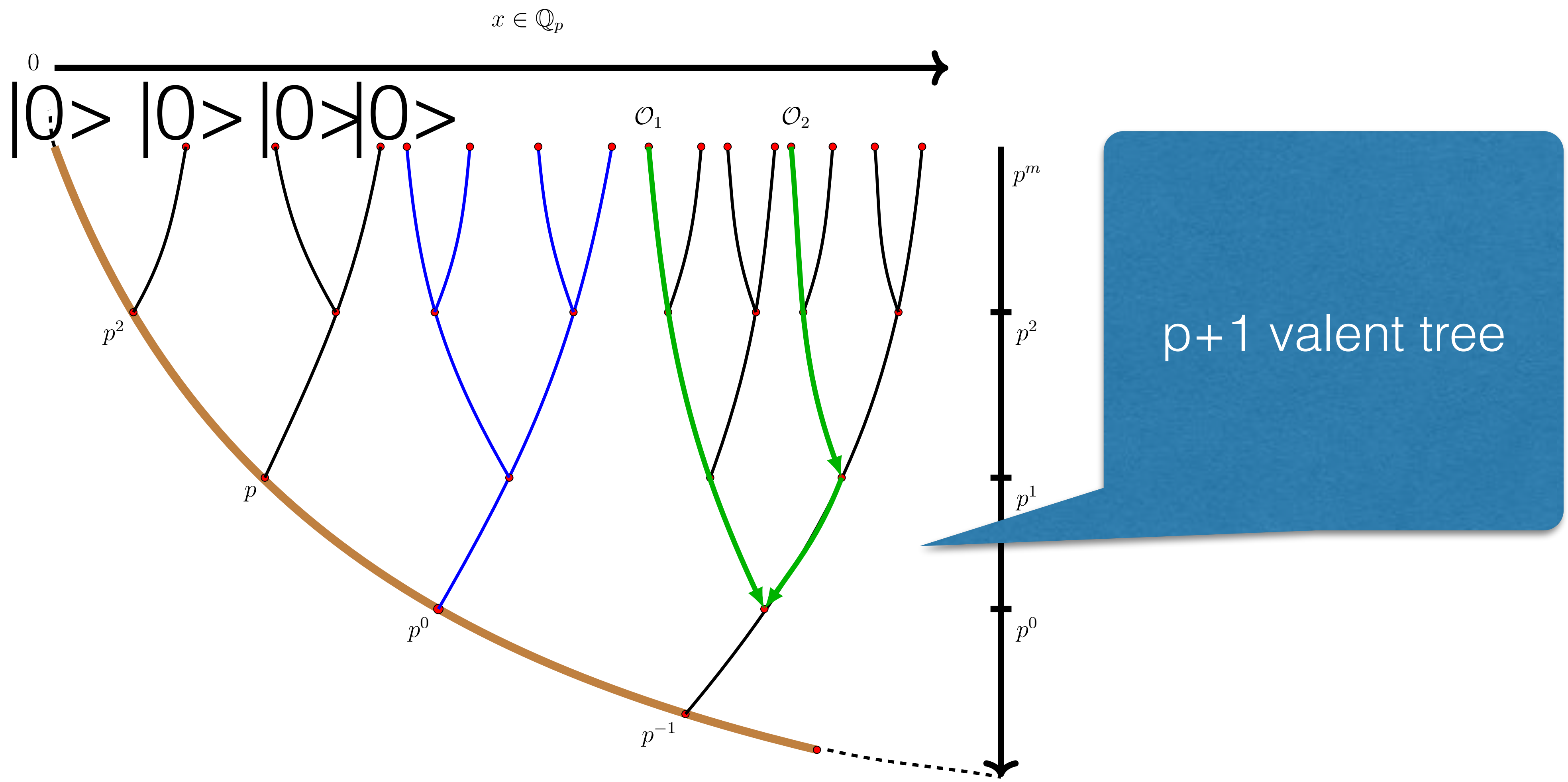
	upper half plane $\mathbb{H}$	Bruhat-Tits tree $\mathbb{H}_p$
Isometry group G	$SL(2,\mathbb{R})$	$PGL(2,\mathbb{Q}_p)$
Isotopy group K	$SO(2,\mathbb{R})$	$PGL(2,\mathbb{Z}_p)$
Boundary	$\mathbb{R}$	$\mathbb{Q}_p$
	$\mathbb{H} = SL(2,\mathbb{R})/SO(2,\mathbb{R})$.	$\mathbb{H}_p \equiv \frac{PGL(2,\mathbb{Q}_p)}{PGL(2,\mathbb{Z}_p)}$

The bulk is discrete.



Putting together the tensor network and the Bruhat-Tits tree: a proprosal for partition function and correlation functions

(tensor network of the partition function), HLY, Li, Melby-Thompson 2019



partition function

Graph Laplacian

$$\Box\phi(v) = \sum_{u\sim v}(\phi(u) - \phi(v))$$

- consider $G(v_1, v_2) = p^{-\Delta d(v_1,v_2)}$
- p+1 = valancy of graph
- We have $$(\Box_{v_1} + m^2)G(v_1, v_2) = \mathcal{N}\delta_{v_1,v_2}$$

$$m^2 = -\frac{1}{\zeta_p(\Delta-1)\zeta(\Delta)}, \qquad \zeta_p(s) \equiv \frac{1}{1-p^{-s}}$$

Putting together the tensor network and the BT tree

the labels of the tensors are primaries of the CFT

p=2 $$G_{I_1 I_2 I_3} = p^{-\Delta_{I_1} - \Delta_{I_2} - \Delta_{I_3}} C_{I_1 I_2 I_3}$$

$$G_{I_1 I_2 I_3 \cdots I_{p+1}} = p^{-\Delta_{I_1} - \Delta_{I_2} - \Delta_{I_3} - \cdots \Delta_{I_{p+1}}} C_{I_1 I_2 I_3 \cdots I_{p+1}}$$

$$C_{I_1 \cdots I_n} = C^{J_1}_{I_1 I_2} C^{J_2}_{J_1 I_3} \cdots C_{J_{n-2} I_{n-1} I_n}$$

(C_{ij1} can be diagonalised)

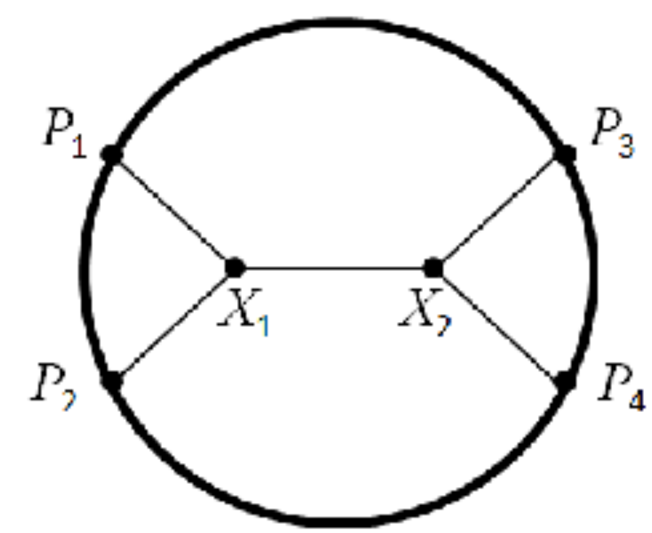
HKLL Relation/Witten Diagram

$$[\Box + m^2]K(x, z|y) = 0$$

$$\Delta = \frac{d}{2} \pm \sqrt{d^2/4 + m^2 L^2}$$

Correlation function

- HKLL relation :
- $\phi(x, z) = \int d^d y K(x, z|y)\mathcal{O}(y)$
-

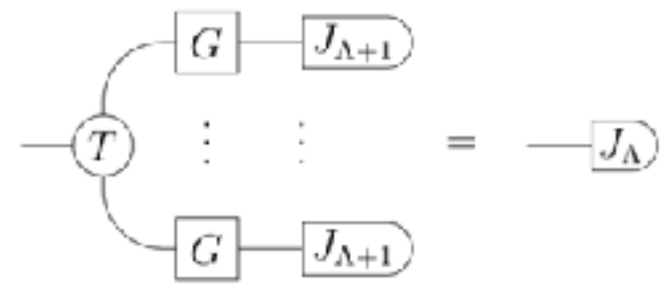


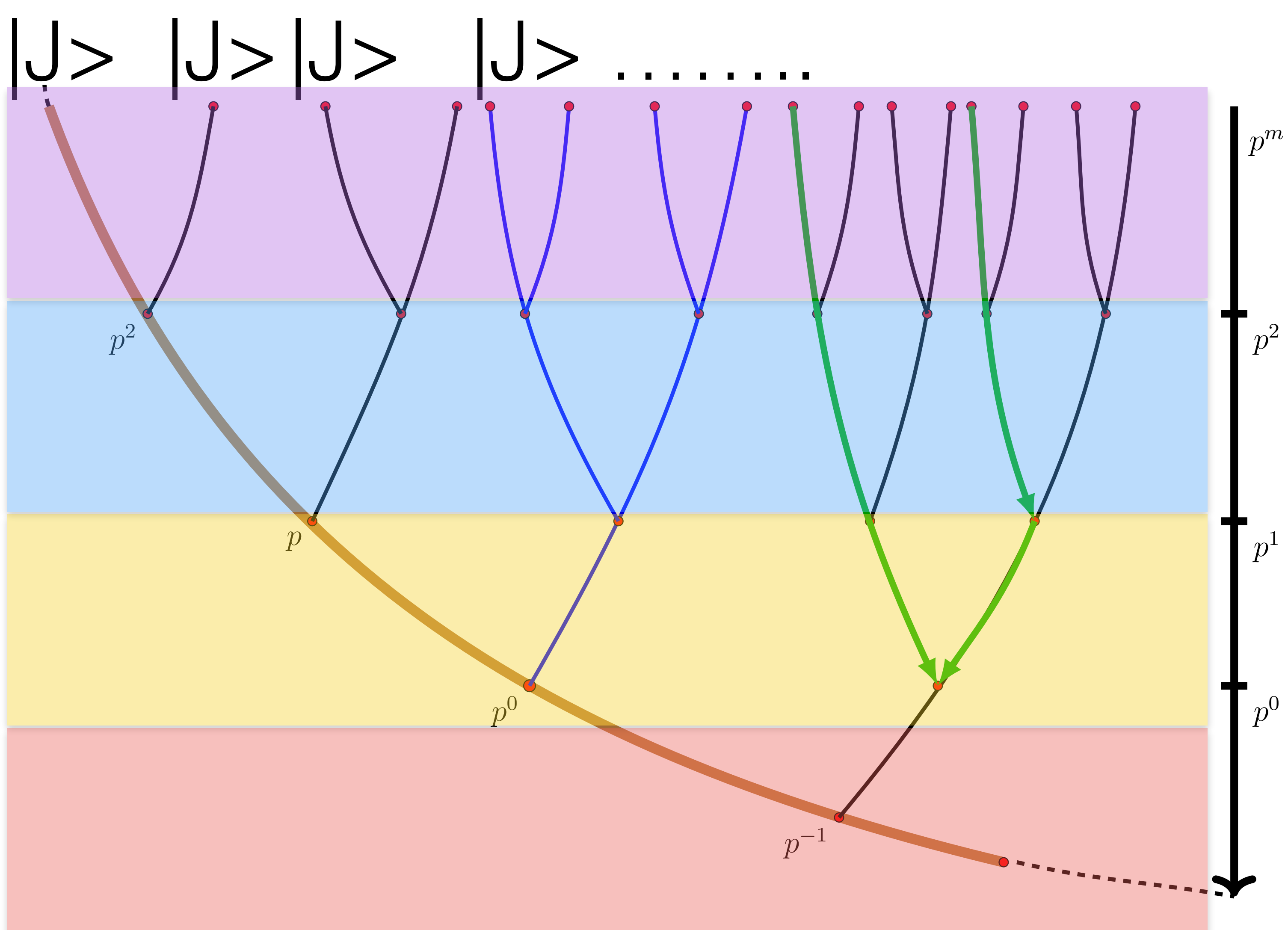
pic courtesy M. Paulos

P-adic RG flow

HLY, Li, Melby-Thompson 2019

Define RG flow $|J_n\rangle = |0\rangle + J_n^a |a\rangle$





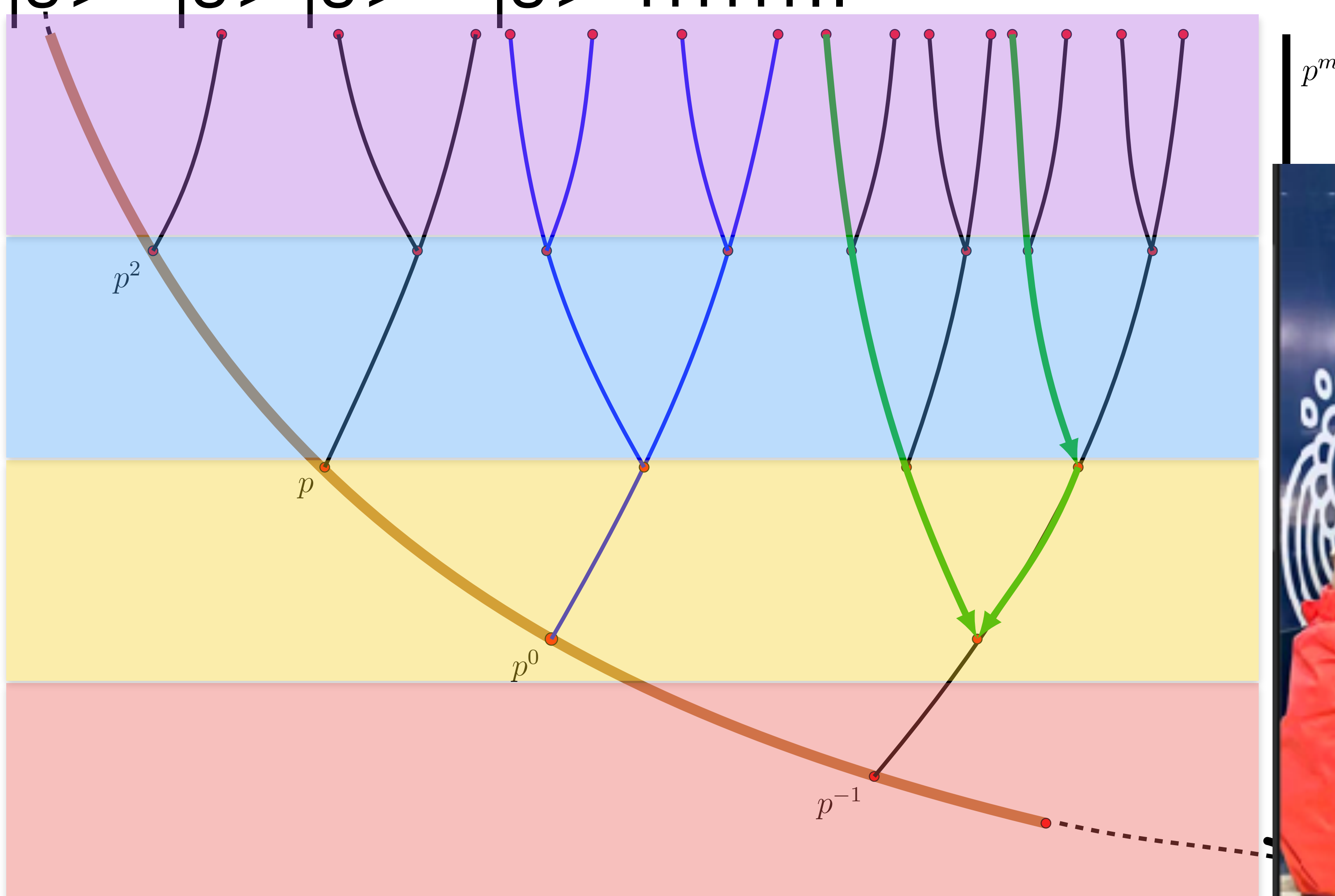
Is there some analogue of "Einstein Equation" that can describe this flow?

Deforming from pure pAdS space

Define RG flow

$$|J_n\rangle = |0\rangle + J_n^a |a\rangle$$

|J> |J> |J> |J> ....



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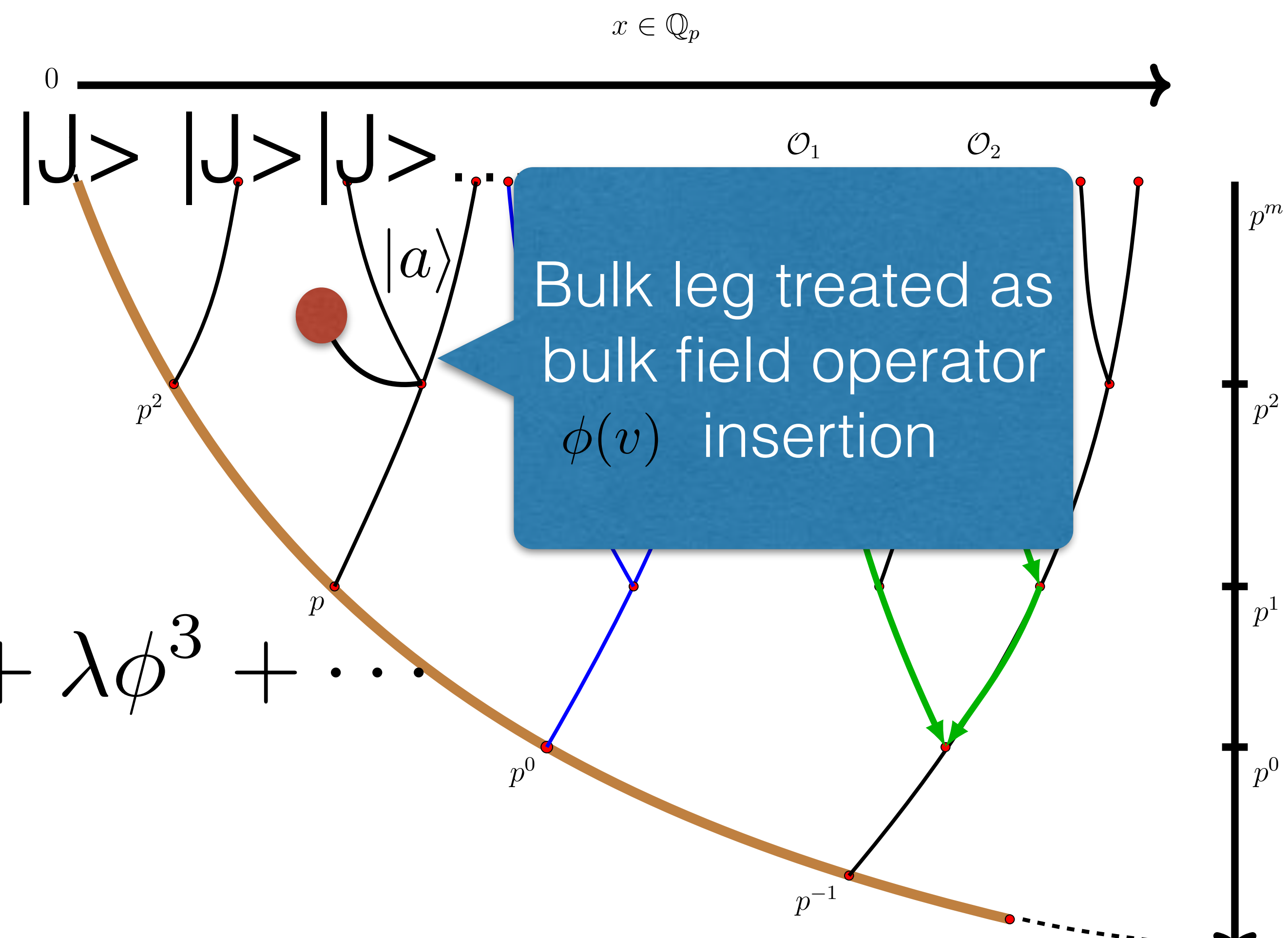
Recipe of Einstein equation (Some Guesses……)

1. Bulk expectation value of some scalar field. Compute stress tensor assuming the simplest possible kinetic term + possible interaction terms…



$$\partial\phi\partial\phi + m^2\phi^2 + \lambda\phi^3 + \cdots$$

$$\langle \phi_a(v) \rangle = \sum_{x_b, a} G^{\Delta_a}(x_b, v) J_a + \frac{1}{2} \sum_{a,b,c} \sum_{x_b, y_b, z} J_b J_c C_{abc} G^{\Delta_b}(x_b, z) G^{\Delta_c}(y_b, z) G^{\Delta_a}(z, v) + \mathcal{O}(J^3)$$

Steps:

- 1) Assign distance to the edges ?
- 2) Assign curvatures to the graph? — then write down Einstein-Hilbert action
- 3) Read off bulk expectation values to bulk fields and write down effective bulk matter action
- Is the Einstein equation satisfied?

This bit we solved in 2019 but need covariantization

Steps:

$$V^a_{xy} = \delta^a_1 + \omega^a_{xy},\ \omega^a_{xy} \equiv \lambda^{(1)\,a}_{xy} + \lambda^{(2)\,a}_{xy} + \cdots, \quad (15)$$

$$\tilde{V}^a_{xy} = \delta^a_1 + \tilde{\omega}^a_{xy},\ \tilde{\omega}^a_{xy} \equiv \tilde{\lambda}^{(1)\,a}_{xy} + \tilde{\lambda}^{(2)\,a}_{xy} + \cdots, \quad (16)$$

- .

$$\begin{aligned} j_e = {} & A^a(\omega^a_e + \tilde{\omega}^a_e) + B^{ab}(\omega^a_e\omega^b_e + \tilde{\omega}^a_e\tilde{\omega}^b_e) + C^{ab}\omega^a_e\tilde{\omega}^b_e \\ & + D^{abc}(\omega^a_e\omega^b_e\omega^c_e + \tilde{\omega}^a_e\tilde{\omega}^b_e\tilde{\omega}^c_e) \\ & + E^{abc}(\omega^a_e\omega^b_e\tilde{\omega}^c_e + \tilde{\omega}^a_e\tilde{\omega}^b_e\omega^c_e) + \mathcal{O}(\omega^4), \end{aligned} \quad (17)$$

- 2) Assigr down Ein

$$R_x = a_0 + a_1 \sum_i j_{xy_i} + b \sum_i j^2_{xy_i} + c \sum_{i \neq k} j_{xy_i} j_{xy_k} + \mathcal{O}(j^3), \quad (18)$$

- 3) Read off bulk expectation and write down effective bul
- Is the Einstein equation satis

$$\begin{aligned} S^{cov}_m &= S^{cov}_2 + S^{cov}_3 + \cdots \\ S^{cov}_2 &= \sum_{\langle xy\rangle} d^k_{\langle xy\rangle}(\phi^a_x - \phi^a_y)^2 + \sum_{\langle xy\rangle} \frac{d_{\langle xy\rangle}}{p+1} m^2_a((\phi^a_x)^2 + (\phi^a_y)^2) \\ S^{cov}_3 &= \sum_{\langle xy\rangle} \Big(h(d_{\langle xy\rangle}) H^{abc}(\phi^a_x\phi^b_x\phi^c_x + \phi^a_y\phi^b_y\phi^c_y) \\ &\quad + r(d_{\langle xy\rangle}) R^{abc}(\phi^a_x\phi^b_x\phi^c_y + \phi^a_y\phi^b_y\phi^c_x) \Big), \end{aligned} \quad (19)$$

Einstein equation and constraints:

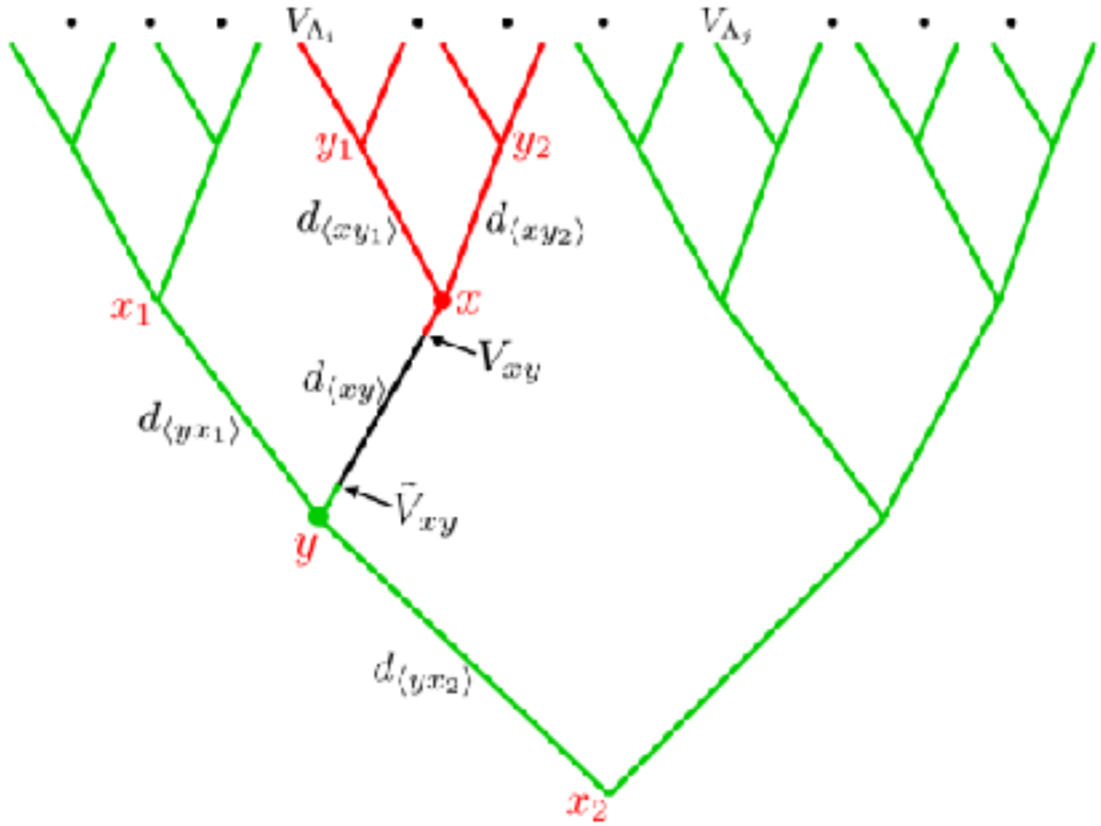
$$G_{xy} \equiv \frac{\delta S_{EH}}{\delta d_{xy}} = \Lambda + 2a_1 + 4bj_{xy}$$
$$+c\Big(\sum_{\substack{i \\ (y_i \neq y)}} j_{xy_i} + \sum_{\substack{i \\ (x_i \neq x)}} j_{x_i y}\Big) + \mathcal{O}(j^2).$$

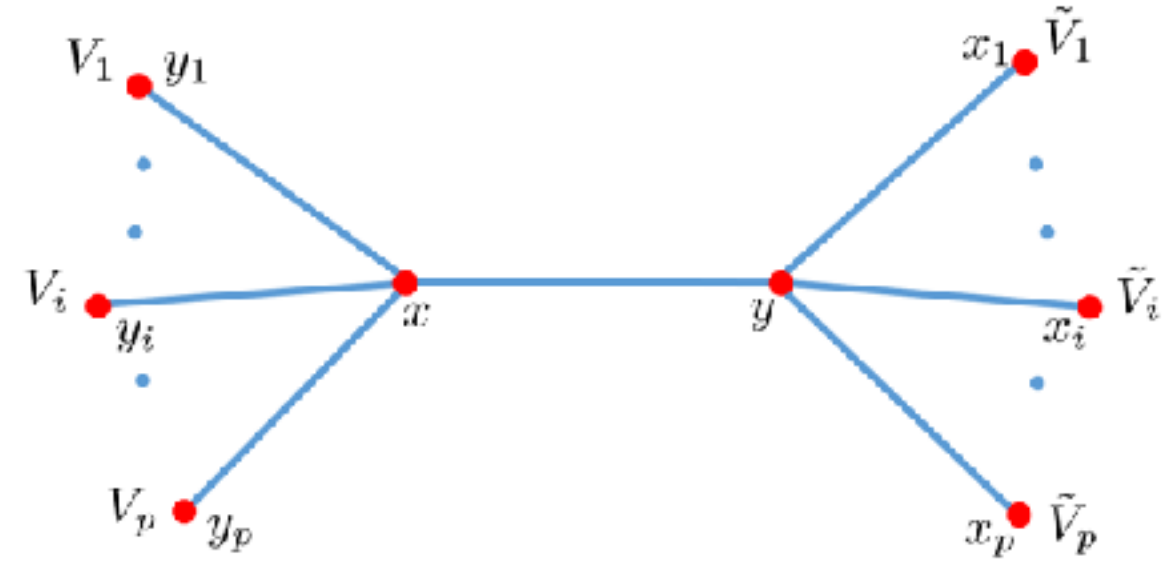
$$G_{xy} + T_{xy} = 0.$$

$$T_{xy} \equiv \frac{\delta S_m^{cov}}{\delta d_{xy}} = \frac{k}{2}(\phi_x^a - \phi_y^a)^2 + \frac{m_a^2((\phi_x^a)^2 + (\phi_y^a)^2)}{2(p+1)}$$
$$+\Big(h_1 H^{abc}(\phi_x^a \phi_x^b \phi_x^c + \phi_y^a \phi_y^b \phi_y^c) + r_1 R^{abc}(\phi_x^a \phi_x^b \phi_y^c$$
$$+\phi_y^a \phi_y^b \phi_x^c)\Big) + \cdots .$$

$$\frac{2b}{c} = -p, \;\; A^a = 0, \;\; \Lambda + 2a_1 = 0.$$

$$m_a^2 = -p - 1 + p^{1-\Delta_a} + p^{\Delta_a},$$
$$k = 1.$$





For the edge xy_i, its V^a and $\tilde{V}^a$ can also be read off from the tensor network:

$$V^a = V_i^a = \delta_1^a + \lambda_i^a + \mathcal{O}(\lambda^2), \tag{4.11}$$

$$\tilde{V}^a = \delta_1^a + (\lambda^a - \lambda_i^a)p^{-\Delta_a} + \tilde{\lambda}^a p^{-2\Delta_a} + \mathcal{O}(\lambda^2). \tag{4.12}$$

Similarly, for the edge $x_i y$, its V^a and $\tilde{V}^a$ are given by

$$V^a = \delta_1^a + (\tilde{\lambda}^a - \tilde{\lambda}_i^a)p^{-\Delta_a} + \lambda^a p^{-2\Delta_a} + \mathcal{O}(\lambda^2), \tag{4.13}$$

$$\tilde{V}^a = \tilde{V}_i^a = \delta_1^a + \tilde{\lambda}_i^a + \mathcal{O}(\lambda^2). \tag{4.14}$$

For the edge xy, its V^a and $\tilde{V}^a$ are given by

$$V^a = \delta_1^a + \lambda^a p^{-\Delta_a} + \mathcal{O}(\lambda^2), \tag{4.15}$$

$$\tilde{V}^a = \delta_1^a + \tilde{\lambda}^a p^{-\Delta_a} + \mathcal{O}(\lambda^2). \tag{4.16}$$

$$G_{xy} = \sum_{a,b} p^{-2(\Delta_a+\Delta_b)} (\lambda^{(1)^a} \lambda^{(1)^b} + \lambda^{\tilde{(1)}^a} \lambda^{\tilde{(1)}^b}) \Big(4bB^{ab} p^{\Delta_a+\Delta_b}$$
$$+ B^{ab} c \left(-2p^{\Delta_a+\Delta_b} + p^{\Delta_a+\Delta_b+1} + p\right) + cC^{ab} p^{2\Delta_a+\Delta_b} \Big)$$
$$+ p^{-2(\Delta_a+\Delta_b)} \lambda^{(1)^a} \lambda^{\tilde{(1)}^b}$$
$$\Big(4bC^{ab} p^{\Delta_a+\Delta_b} + 2B^{ab} c(p-1)\left(p^{\Delta_b} + p^{\Delta_a}\right)$$
$$+ cC^{ab}\left(p^{2\Delta_a} + p^{2\Delta_b}\right) \Big) + \sum_i \frac{c(\lambda^{(1)}{}^a_i \lambda^{(1)}{}^b_i + \lambda^{\tilde{(1)}}{}^a_i \lambda^{\tilde{(1)}}{}^b_i)}{2}$$
$$\Big(2B^{ab}(1 + p^{-\Delta_a-\Delta_b}) - C^{ab}(p^{-\Delta_a} + p^{-\Delta_b}) \Big)$$
$$+ \mathcal{O}(\lambda^3). \tag{A12}$$

$$T_{xy} = \sum_a p^{-3\Delta_a} (\lambda^{(1)^a} \lambda^{(1)^a} + \lambda^{\tilde{(1)}^a} \lambda^{\tilde{(1)}^a})$$
$$\frac{\Big(k(p+1)\Big(p^{\Delta_a} - 1\Big)^2 + m_a^2\left(p^{2\Delta_a} + 1\right) \Big)}{2(p+1)\left(p^{\Delta_a} - 1\right)\left(p^{\Delta_a} + \right)}$$
$$+ \frac{p^{-3\Delta_a}\Big(2m_a^2 p^{\Delta_a} - k(p+1)\left(p^{\Delta_a} - 1\right)^2 \Big)}{(p+1)\left(p^{\Delta_a} - 1\right)\left(p^{\Delta_a} + 1\right)} \lambda^{(1)^a} \lambda^{\tilde{(1)}^a}$$
$$+ \mathcal{O}(\lambda^3). \tag{A13}$$

$$\frac{c}{2}\Big(2B^{ab}(1+p^{-\Delta_a-\Delta_b}) - C^{ab}(p^{-\Delta_a} + p^{-\Delta_b}) \Big) = 0, \tag{A14}$$

w

$$D^{abc} = \frac{-\tilde{C}^{abc} p^{-\Delta_a-\Delta_b-\Delta_c}}{12c(p+1)\left(p^{2\Delta_a} - 1\right)\left(p^{2\Delta_b} - 1\right)\left(p^{2\Delta_c} - 1\right)}$$
$$\Big(-3p^{\Delta_a+\Delta_b+\Delta_c} - 3p^{2(\Delta_a+\Delta_b+\Delta_c)}$$
$$+ p^{3\Delta_a+\Delta_b+\Delta_c} + p^{\Delta_a+3\Delta_b+\Delta_c} + p^{\Delta_a+\Delta_b+3\Delta_c}$$
$$+ p^{2(\Delta_a+\Delta_b)} + p^{2(\Delta_a+\Delta_c)} + p^{2(\Delta_b+\Delta_c)} \Big), \tag{A17}$$

$$H^{abc} = \frac{\tilde{C}^{abc} p^{-\Delta_a-\Delta_b-\Delta_c}\Big(p^{\Delta_a+\Delta_b+\Delta_c} + p \Big)}{12h_1(p+1)\left(p^{2\Delta_a} - 1\right)\left(p^{2\Delta_b} - 1\right)\left(p^{2\Delta_c} - 1\right)}$$
$$\Big(-3p^{\Delta_a+\Delta_b+\Delta_c} - 3p^{2\left(\Delta_a+\Delta_b+\Delta_c\right)} + p^{3\Delta_a+\Delta_b+\Delta_c}$$
$$+ p^{\Delta_a+3\Delta_b+\Delta_c} + p^{\Delta_a+\Delta_b+3\Delta_c} + p^{2(\Delta_a+\Delta_b)}$$
$$+ p^{2(\Delta_a+\Delta_c)} + p^{2(\Delta_b+\Delta_c)} \Big). \tag{A18}$$

Einstein equation and constraints:

$$G_{xy} \equiv \frac{\delta S_{EH}}{\delta d_{xy}} = \Lambda + 2a_1 + 4bj_{xy} + c\Big(\sum_{\substack{i \\ (y_i \neq y)}} j_{xy_i} + \sum_{\substack{i \\ (x_i \neq x)}} j_{x_i y}\Big) + \mathcal{O}(j^2).$$

$$G_{xy} + T_{xy} = 0.$$

reproduces the same perturbative result in Lin, Lu Yau 2011; Gubser et al 2016;

$$T_{xy} \equiv \frac{\delta S_m^{cov}}{\delta d_{xy}} = \frac{k}{2}(\phi_x^a - \phi_y^a)^2 + \frac{m_a^2((\phi_x^a)^2 + (\phi_y^a)^2)}{2(p+1)} + \Big(h_1 H^{abc}(\phi_x^a\phi_x^b\phi_x^c + \phi_y^a\phi_y^b\phi_y^c) + r_1 R^{abc}(\phi_x^a\phi_x^b\phi_y^c + \phi_y^a\phi_y^b\phi_x^c)\Big) + \cdots .$$

$$\frac{2b}{c} = -p, \; A^a = 0, \; \Lambda + 2a_1 = 0. \longrightarrow G_{xy} = -c\Box j_{xy}.$$

$$m_a^2 = -p - 1 + p^{1-\Delta_a} + p^{\Delta_a}, \quad k = 1.$$

When the dust settles — the distance IS a FISHER INFORMATION METRIC!

$$d_{\langle xy \rangle} = 1 - \langle u_x | u_y \rangle,$$

$$|u_x\rangle = \frac{1}{\sqrt{2(p+1)}} \sum_a \tilde{\phi}_x^a |a\rangle.$$

Summary and Outlook

- We have got a toy CFT where the tensor network reconstruction is exact
- We recover a discrete Einstein equation out of it — a first quantitative result from tensor network reconstructions
- 1 tensor network — many worlds — but probably not all possible worlds..
- roadmap to real CFTs ? make use of the PEP tensor construction of topological order

Outlook

- Our result is some discretised realisation of Sung-Sik's new paper. arXiv:2009.11880

- It is suggested that the path-integral of a d-dimensional field theory can be thought of as the overlap of two wave functions in d+1 dimensions.

$$Z = \langle \mathbb{I} | S \rangle$$

- The identity being a state invariant under RG, so that one could evolve the bra with RG flow operator H, but then group it with S that leads to flow of the couplings — this is
very close in spirit to the strange correlator holographic network that we studied here.

Outlook

- Quantitative control of descendants which could allow control of sub-AdS locality and gravitational excitations (?)
  You, Milsted, Vidal 2018, 2020; You, Vidal 2020

- Generalization to higher dimensions, and Categorical symmetry
  Verstraete et al ; Gaiotto, Kulp 20;
  Kong, Zheng 2017; Ji, Wen 2019; Kong, Lan, Wen, Zhang, Zheng 2020 …..

Thank you very much!